

CONTINUITY OF SPECTRAL AVERAGING

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ABSTRACT. We consider averages κ of spectral measures of rank one perturbations with respect to a σ -finite measure ν . It is examined how various degrees of continuity of ν with respect to α -dimensional Hausdorff measures ($0 \leq \alpha \leq 1$) are inherited by κ . This extends Kotani's trick where ν is simply the Lebesgue measure.

1. INTRODUCTION

Let A be a bounded self adjoint operator on a separable Hilbert space \mathcal{H} . Fix a normalized vector $\phi \in \mathcal{H}$. Consider the family of rank one perturbations

$$(1.1) \quad A_\lambda := A + \lambda \langle \phi, \cdot \rangle \phi ,$$

indexed by the real parameter λ . Despite its simple form, the family in (1.1) proves to be a very useful tool in the study of discrete random Schrödinger operators. There, rank one perturbations correspond to fluctuations of the potential at a lattice site. Ref. [1] summarizes several of these applications, among them the Simon-Wolf criterion for spectral localization, the theory of Aizenman-Molchanov for the Anderson model, and Wegner's estimate.

Crucial to most of these applications is a result known as spectral averaging or Kotani's trick. It allows to relate the spectral behavior for fixed values of λ to the spectral properties inherent to the entire family $\{A_\lambda\}$, i.e. upon a variation of λ .

Denote by $d\mu(x)$ and $d\mu_\lambda(x)$ the spectral measure with respect to ϕ for the operator A and A_λ , respectively. Kotani's trick is the following result:

Theorem 1.1 (Kotani's trick). *Let B be a Borel set on the real line. Then,*

$$(1.2) \quad |B| = \int \mu_\lambda(B) d\lambda .$$

Here, $|\cdot|$ denotes the Lebesgue measure.

Different proofs and applications of this result were given in [2, 3, 4, 5, 6, 7, 8, 9]. We note that for some purposes, among them the Simon-Wolf criterion, a weaker formulation is sufficient; this weaker result states that the Borel measure on the right hand side of (1.2) is absolutely continuous with respect to Lebesgue. In fact, in the original proof of the Simon-Wolf criterion (see [9], Theorem 5) the authors show that the measure

$$(1.3) \quad \kappa(\cdot) = \int \mu_\lambda(\cdot) \frac{1}{1 + \lambda^2} d\lambda ,$$

is mutually equivalent to the Lebesgue measure.

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Eq. (1.3) suggests the following generalization: For ν a σ -finite Borel measure on \mathbb{R} , define a measure κ by

$$(1.4) \quad \kappa(\cdot) = \int \mu_\lambda(\cdot) d\nu(\lambda) .$$

Such averages were first considered in [10] for a finite measure ν . There, relation (3.4) was discovered for a finite measure ν and used to estimate the Hausdorff dimension of set $\{\lambda : A_\lambda \text{ has some continuous spectrum}\}$ (see Theorem 5.2 therein).

In view of the measure defined in (1.4), Kotani's trick ($d\nu(\lambda) = d\lambda$) and the result for $d\nu(\lambda) = \frac{1}{1+\lambda^2}d\lambda$ in (1.3) become statements about continuity properties of the measure ν being inherited by κ .

In this article we pursue this continuity based approach to spectral averaging. We will show how various degrees of continuity of ν with respect to α -dimensional Hausdorff measures ($\alpha \leq 1$) are inherited by κ . For a definition of α continuity see definition 4.1. Our main result is the following Theorem:

Theorem 1.2. *If ν is absolutely continuous with respect to Lebesgue, so is κ . Additionally, if ν is αc , $0 < \alpha < 1$, then κ is δc for all $\delta < \alpha$.*

Kotani's trick then arises as a special case, where the density of $d\kappa(x)$ can be calculated explicitly.

The paper is organized as follows: Sec. 2 summarizes some results of the theory of Borel transforms and rank one perturbations as needed for the further development. After showing that mere continuity of ν is inherited by κ (Theorem 3.1), we examine the situation for ν being uniformly- α -Hölder continuous (see definition 3.3). In particular, we shall show that uniform 1-Hölder continuity of ν is inherited by κ . Kotani's trick follows as special case if $d\nu(x) = dx$. Finally, in Sec. 4 and 5 we employ the Rogers-Taylor decomposition of measures with respect to Hausdorff measures to prove theorem 1.2.

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2. BOREL TRANSFORMS & RANK ONE PERTURBATIONS

The key quantity to understand the spectral properties of the family (1.1) is the Borel transform of the spectral measure $d\mu$ associated with the *unperturbed* operator A and the vector ϕ . In general, if η is a Borel measure with $\int \frac{1}{1+|y|} d\eta(y) < \infty$, for $z \in \mathbb{H}^+$ we define

$$(2.1) \quad F_\eta(z) := \int \frac{d\eta(y)}{y - z} ,$$

the Borel transform of the measure η . Letting $z = x + i\epsilon$, $\epsilon > 0$, we may split $F_\eta(x + i\epsilon)$ into its real and imaginary part, i.e. $F_\eta(x + i\epsilon) =: Q_\eta(x + i\epsilon) + iP_\eta(x + i\epsilon)$, where

$$\begin{aligned} Q_\eta(x + i\epsilon) &= \int \frac{y - x}{(y - x)^2 + \epsilon^2} d\eta(y) , \\ P_\eta(x + i\epsilon) &= \int \frac{\epsilon}{(y - x)^2 + \epsilon^2} d\eta(y) . \end{aligned}$$

We shall refer to P_η and Q_η as Poisson- and conjugate Poisson transform of the measure η , respectively. Whereas $Q_\eta(x + i\epsilon)$ depends on the “symmetry” of η

around x , $P_\eta(x + i\epsilon)$ carries information about the growth of the measure η at x . A detailed analysis about the asymptotic behavior of P_η and Q_η is given in [11].

The relation between the local growth of a measure and its Poisson transform follows from the following simple estimate: Given $\alpha \in [0, 1]$, then for $x \in \mathbb{R}$ and $\epsilon > 0$

$$(2.2) \quad \epsilon^{1-\alpha} P_\eta(x + i\epsilon) \geq \epsilon^{1-\alpha} \int_{(x-\epsilon, x+\epsilon)} \frac{\epsilon}{(x-y)^2 + \epsilon^2} d\eta(y) \geq \frac{1}{2\epsilon^\alpha} M_\eta(x; \epsilon),$$

where $M_\eta(x; \epsilon) := \eta(x - \epsilon, x + \epsilon)$ denotes the growth function of η at x .

Remark 2.1. As will be seen below (see Theorem 2.2), it is useful to consider the Poisson transform of a measure even if its Borel transform does not exist. A necessary and sufficient condition for $P_\eta(x + i\epsilon) < \infty$, $\epsilon > 0$, is $\int \frac{1}{1+x^2} d\eta(x) < \infty$.

For $\alpha \geq 0$ we define,

$$(2.3) \quad \overline{D}_\eta^\alpha(x) := \limsup_{\epsilon \rightarrow 0^+} \frac{\eta(x - \epsilon, x + \epsilon)}{\epsilon^\alpha},$$

the upper- α -derivative of a measure η at a point $x \in \mathbb{R}$.

Above estimate (2.2) leads to the following result proven e.g. in [10]:

Proposition 2.1. *Let $\alpha \in [0, 1]$ and $x \in \mathbb{R}$ be fixed. Then $\overline{D}_\eta^\alpha(x)$ and $\limsup_{\epsilon \rightarrow 0^+} \epsilon^{1-\alpha} P_\eta(x + i\epsilon)$ are either both infinite, zero, or in $(0, +\infty)$.*

Proposition 2.1 will be used to analyze continuity with respect to Hausdorff measures of the measure κ defined in (1.4).

The following Theorem is key to spectral analysis of rank one perturbations. It provides a characterization of the components of η in a Lebesgue decomposition. Proof can be found e.g. in [1, 12].

Theorem 2.2. *Let η be a Borel measure on the real line such that $\int \frac{1}{1+y^2} d\eta(y) < \infty$. The following statements characterize the components in the Lebesgue decomposition of $\eta = \eta_{\text{sing}} + \eta_{\text{ac}}$:*

- (i) $d\eta_{\text{ac}}(x) = \frac{1}{\pi} P_\eta(x + i0) dx$.
- (ii) η_{sing} is supported on $\{x : P_\eta(x + i0) = +\infty\}$.

Theorem 2.2 implies a characterization of the spectral properties of the family $\{A_\lambda\}$. Out of this we shall only need the following statement related to the singular(pp+sc)-spectrum of $\{A_\lambda\}$ (see [1, Theorem 12.2])

Proposition 2.2. (i) $\mu_{\lambda, \text{sing}}$ is supported on the set $\{x : F_\mu(x + i0) = -\frac{1}{\lambda}\}$.
(ii) The family of measures $\{d\mu_{\lambda, \text{sing}}\}$ are mutually singular. In particular, a point $x \in \mathbb{R}$ can be an atom for at most one value of λ .

3. SPECTRAL AVERAGING

For a fixed σ -finite Borel measure ν , consider the measure κ introduced in (1.4). κ is well defined since for any polynomial $p(x)$, $\langle \phi, p(A_\lambda) \phi \rangle$ is a polynomial in λ . Stone-Weierstraß and functional calculus then imply that $\lambda \mapsto \mu_\lambda(B)$ is Borel measurable for any Borel set $B \subseteq \mathbb{R}$.

We start our analysis of the continuity of κ in relation to the continuity of ν with the following simple observation:

Theorem 3.1. *If ν is continuous, so is κ .*

Proof. Apply part (ii) of Proposition 2.2 to $\kappa(\{x\}) = \int \mu_\lambda(\{x\}) d\nu(\lambda)$, $x \in \mathbb{R}$. \square

The following simple relation between the Poisson transforms of κ and ν is crucial to further analyze the continuity properties of κ .

Proposition 3.1. *Assume $\int \frac{1}{1+y^2} d\nu(y) < \infty$. Then,*

$$(3.1) \quad P_\kappa(z) = P_\nu \left(-\frac{1}{F_\mu(z)} \right),$$

for $z \in \mathbb{H}^+$.

Proof. Using the definition of κ in (1.4), the monotone convergence Theorem implies

$$\int f(x) d\kappa(x) = \int \left\{ \int f(x) d\mu_\lambda(x) \right\} d\nu(\lambda),$$

for any measurable $0 \leq f$.

In particular for $z \in \mathbb{H}^+$,

$$(3.2) \quad \begin{aligned} P_\kappa(z) &= \int P_{\mu_\lambda}(z) d\nu(\lambda) \\ &= \int \frac{P_\mu(z)}{|1 + \lambda F_\mu(z)|^2} d\nu(\lambda) = P_\nu \left(-\frac{1}{F_\mu(z)} \right). \end{aligned}$$

Here, the second equality follows from the Aronszajn-Krein formula [1]

$$(3.3) \quad F_{\mu_\lambda}(z) = \frac{F_\mu(z)}{1 + \lambda F_\mu(z)},$$

which relates the Borel transforms of the spectral measures μ_λ and μ . \square

Remark 3.2. If ν is a finite measure an analogous result between the respective Borel transforms was first obtained in [10]:

$$(3.4) \quad F_\kappa(z) = F_\nu \left(-\frac{1}{F_\mu(z)} \right)$$

Note that for non-finite ν the Borel transform will in general not exist (e.g. take ν to be the Lebesgue measure). In fact for σ -finite ν , often the Poisson transform exists whereas its Borel transform does not. In these cases we still have a relation between the Poisson transforms of ν and κ as established in Proposition 3.1.

In order to prove finer statements on the continuity of κ , we first establish some results for uniformly Hölder continuous ν . Recall the following definition:

Definition 3.3. Let η be a σ -finite Borel measure on the real line and $\alpha \geq 0$. η is uniformly α Hölder continuous (U α H) if for some constant K , $\eta(I) \leq K|I|^\alpha$ for any interval I .

Remark 3.4. (i) U1H implies absolute continuity.

(ii) Using the Rogers-Taylor decomposition Theorem (see Theorem 4.2), there are no *non-trivial* U α H measures for $\alpha > 1$.

For ν U α H, Proposition 3.1 implies the following key estimate for the Poisson transform of κ :

Proposition 3.2. *If ν is $U\alpha H$, $0 \leq \alpha \leq 1$, then for some constant C_α and all $z \in \mathbb{H}^+$*

$$(3.5) \quad P_\kappa(z) \leq C_\alpha \left(\frac{|F_\mu(z)|^2}{P_\mu(z)} \right)^{1-\alpha}.$$

In particular, $\int \frac{1}{1+x^2} d\kappa(x) < \infty$, whence κ is a locally finite Borel measure (i.e. finite on compact sets).

Proof. Let $z \in \mathbb{H}^+$. Recasting $P_\nu(z)$ in terms of the Lebesgue-Stieltjes measure induced by $M_\nu(\text{Re}\{z\}; \delta)$, we get

$$(3.6) \quad \begin{aligned} P_\nu(z) &= \text{Im}\{z\} \int_0^{+\infty} \frac{dM_\nu(\text{Re}\{z\}; \delta)}{\delta^2 + \text{Im}\{z\}^2} \\ &= \text{Im}\{z\} \int_0^{+\infty} 2\delta \frac{M_\nu(\text{Re}\{z\}; \delta)}{(\delta^2 + \text{Im}\{z\}^2)^2} d\delta \leq \text{Im}\{z\} 2K \int_0^{+\infty} \frac{\delta^{\alpha+1}}{(\delta^2 + \text{Im}\{z\}^2)^2} d\delta \\ &= \frac{\pi\alpha K}{2 \sin(\frac{\pi\alpha}{2})} (\text{Im}\{z\})^{\alpha-1}. \end{aligned}$$

Here, the second equality follows using integration by parts; the last equality is obtained by contour integration. For $\alpha = 0$, the last equality in (3.6) is to be interpreted in the limit $\alpha \rightarrow 0$, i.e. $P_\nu(z) \leq K \text{Im}\{z\}^{-1}$.

In particular, for $\nu \text{ } U\alpha H$, (3.6) establishes $\int \frac{1}{1+x^2} d\nu(x) < \infty$. Application of Proposition 3.1 hence yields the desired estimate. \square

Proposition 3.2 reveals the special nature of the case $\alpha = 1$. Then, we obtain a *uniform* upper bound for the Poisson transform of κ valid in all of \mathbb{H}^+ . In fact, this implies for κ to inherit “full” continuity of the measure ν :

Theorem 3.5. *If ν is $U1H$, so is κ .*

Proof. Let $0 \leq f$ be continuous of compact support. Using Proposition 3.2,

$$(3.7) \quad \begin{aligned} \int f(x) dx &\geq \limsup_{\epsilon \rightarrow 0^+} \frac{1}{C_1} \int f(x) P_\kappa(x + i\epsilon) dx \\ &= \limsup_{\epsilon \rightarrow 0^+} \frac{1}{C_1} \int \left(\int f(x) \frac{\epsilon}{(x-y)^2 + \epsilon^2} dx \right) d\kappa(y) \\ &\geq \frac{1}{C_1} \int \left(\lim_{\epsilon \rightarrow 0^+} \int f(x) \frac{\epsilon}{(x-y)^2 + \epsilon^2} dx \right) d\kappa(y) \\ &= \frac{\pi}{C_1} \int f(y) d\kappa(y). \end{aligned}$$

Here, the second equality follows from Tonelli, whereas the second inequality uses Fatou’s Lemma. Note that σ -finiteness of κ is implied by Proposition 3.2. \square

Theorem 3.5 in particular implies $d\kappa(x) \ll dx$. Spectral averaging now arises as a special case where the density of κ can be calculated explicitly.

Proof of Theorem 1.1. Since the Poisson transform of the Lebesgue measure $P_{\text{Leb}}(z) = \pi$, all $z \in \mathbb{H}^+$, Theorem 2.2(i) and Proposition 3.1 yield $d\kappa(x) = dx$. \square

4. CONTINUITY WITH RESPECT TO HAUSDORFF MEASURES

In this section we analyze the degree of continuity of κ induced by measures ν with lesser degree of continuity than considered in the previous section. To this end we make the following definitions:

Definition 4.1. For $0 \leq \alpha$ let h^α denote the α -dimensional Hausdorff measure on \mathbb{R} . Let η be a Borel measure on the real line.

- (1) η is called α -continuous (αc) if $\eta(B) = 0$ whenever $h^\alpha(B) = 0$.
- (2) η is called α -singular if η is supported on a set of zero measure h^α .

The main tools for proving Theorem 1.2 are the following two results due to Rogers & Taylor [13, 14, 15]:

Theorem 4.2 (Rogers & Taylor - 1 (see Theorem 67 in [13])). *Let η be a σ -finite Borel measure on \mathbb{R} and $0 \leq \alpha \leq 1$. Consider the sets $T_{\eta;0+}^\alpha := \{x : 0 \leq \overline{D}_\eta^\alpha(x) < \infty\}$ and $T_{\eta;\infty}^\alpha := \{x : \overline{D}_\eta^\alpha(x) = \infty\}$. Then, $T_{\eta;0+}^\alpha$ and $T_{\eta;\infty}^\alpha$ are Borel measurable and*

- (i) η is αc on $T_{\eta;0+}^\alpha$,
- (ii) $h^\alpha(T_{\eta;\infty}^\alpha) = 0$ and η is α -singular on $T_{\eta;\infty}^\alpha$.

Theorem 4.3 (Rogers & Taylor - 2 (see Theorem 68 in [13])). *Let η be σ -finite and α -continuous, $\alpha \geq 0$. For $\epsilon > 0$, there exist mutually singular measures η_1 and η_2 with $\eta = \eta_1 + \eta_2$ such that*

- (i) η_1 is $U\alpha H$ and
- (ii) $\eta_2(\mathbb{R}) < \epsilon$.

Remark 4.4. Depending on \overline{D}_η^α , Theorem 4.2 decomposes η into an αc and an α -singular component. It thus generalizes the usual Lebesgue decomposition for $\alpha = 1$. The relevance of the Rogers Taylor decomposition in spectral theory was pointed out by Last, see [16].

By Theorem 4.3, any α -continuous measure is almost $U\alpha H$. Hence, the proof of Theorem 1.2 boils down to establishing the statement for a $U\alpha H$ measure ν . To this end we shall use the following Lemma, which quantifies the asymptotic growth of P_η and Q_η near the support of a probability measure η .

Lemma 4.5. *Let η be a probability (Borel) measure on \mathbb{R} , then for $x \in \mathbb{R}$ and $\epsilon > 0$*

$$\max\{P_\eta(x + i\epsilon), |Q_\eta(x + i\epsilon)|\} \leq \frac{2}{\epsilon} \sum_{n=0}^{\infty} 2^{-n} M_\eta(x; 2^{n+1}\epsilon) .$$

Proof. Let $x \in \mathbb{R}$ and $\epsilon > 0$.

$$\begin{aligned} |Q_\eta(x + i\epsilon)| &\leq \sum_{n=1}^{\infty} \int_{\epsilon 2^n \leq |x-y| \leq \epsilon 2^{n+1}} \frac{|x-y|}{(x-y)^2 + \epsilon^2} d\eta(y) \\ &+ \int_{|x-y| \leq 2\epsilon} \frac{|x-y|}{(x-y)^2 + \epsilon^2} d\eta(y) \leq \frac{2}{\epsilon} \sum_{n=0}^{\infty} 2^{-n} M_\eta(x; 2^{n+1}\epsilon) . \end{aligned}$$

By a similar computation we obtain the same upper bound for $P_\eta(x + i\epsilon)$. \square

We note that this result is an extended version of a Lemma in [10] for $U\alpha H$ η .

Together with Proposition 3.2, Lemma 4.5 allows to control the asymptotic behaviour of $P_\kappa(x + i\epsilon)$ as $\epsilon \rightarrow 0^+$. We are thus in the position to prove Theorem 1.2.

5. PROOF OF THE MAIN THEOREM (THEOREM 1.2)

We shall divide the proof into two steps; step 1 establishes the statement for ν $U\alpha H$. Theorem 4.3 then allows to extend the result to the αc case (step 2).

Step 1: Assume ν to be $U\alpha H$. If $\alpha = 1$, the statement follows directly from Theorem 3.5. Let $\alpha < 1$. We first examine the situation outside the support of the measure μ .

Proposition 5.1. *Let $0 < \alpha < 1$ and ν $U\alpha H$. Then κ is αc outside $\text{supp}\mu$.*

Proof. Fixing $x \notin \text{supp}\mu$, there exist positive constants Γ_1 and Γ_2 such that

$$|F_\mu(x + i\epsilon)|^2 \leq \Gamma_1, \quad P_\mu(x + i\epsilon) \geq \Gamma_2\epsilon,$$

for all $\epsilon > 0$ sufficiently small. Hence by Proposition 3.2 we obtain,

$$(5.1) \quad \epsilon^{1-\alpha} P_\kappa(x + i\epsilon) \leq C_\alpha \epsilon^{1-\alpha} \left(\frac{|F_\mu(z)|^2}{P_\mu(z)} \right)^{1-\alpha} \leq C_\alpha \left(\frac{\Gamma_1}{\Gamma_2} \right)^{1-\alpha},$$

which implies the claim by Theorem 4.2. \square

Remark 5.1. By Theorem 4.3 (see the argument given in step 2), the statement of Proposition 5.1 remains valid if ν is (only) αc .

In order to analyze the situation within the support of μ , we first establish the following Lemma:

Lemma 5.2. *Let $0 < \alpha < 1$ and ν $U\alpha H$. Fix $0 < \beta < 1$. Then, κ is γc on the set $T_{\mu;0+}^\beta$ where*

$$(5.2) \quad \gamma(\alpha, \beta) = \alpha - 2(1 - \beta)(1 - \alpha),$$

as long as $\beta > \max\left\{0, \frac{2-3\alpha}{2(1-\alpha)}\right\}$.

Proof. Let $\beta < 1$ be fixed. By Proposition 5.1 the statement is true outside $\text{supp}\mu$. Let $x \in \text{supp}\mu$ and assume $\overline{D}_\mu^\beta(x) < \infty$ so that $M_\mu(x; \delta) \leq \Lambda_x \delta^\beta$, $\forall \delta > 0$. Thus,

$$(5.3) \quad \frac{2}{\epsilon} \sum_{n=0}^{\infty} 2^{-n} M_\mu(x; 2^{n+1}\epsilon) \leq \Lambda_x 2^{1+\beta} \epsilon^{\beta-1} \sum_{n=0}^{\infty} 2^{-n(1-\beta)} < \infty.$$

Note that finiteness of the upper bound in (5.3) requires $\beta < 1$.

Let $\gamma < 1$. Using Proposition 3.2 and Lemma 4.5, estimate (5.3) yields

$$(5.4) \quad \epsilon^{1-\gamma} P_\kappa(x + i\epsilon) \leq B_{x,\beta} \left(\frac{\epsilon^{2(\beta-1) + \frac{1-\gamma}{1-\alpha}}}{P_\mu(x + i\epsilon)} \right)^{1-\alpha}.$$

By Theorem 4.2 and Proposition 2.1, κ will be γc on the set $\{x : \limsup_{\epsilon \rightarrow 0^+} \epsilon^{1-\gamma} P_\kappa(x + i\epsilon) < \infty\}$. Choose γ such that $2(\beta - 1) + \frac{1-\gamma}{1-\alpha} = 1$, i.e. $\gamma = \alpha - 2(1 - \beta)(1 - \alpha)$. Since, $\epsilon^{-1} P_\mu(x + i\epsilon) \rightarrow \int \frac{1}{(x-y)^2} d\mu(y)$ as $\epsilon \rightarrow 0^+$ and $\int \frac{1}{(x-y)^2} d\mu(y) > 0$ for $x \in \text{supp}\mu$, we obtain that κ is γc on the set $T_{\mu;0+}^\beta$ with γ determined by (5.2). Finally, $\gamma > 0$ is ensured by requiring $\beta > \max\left\{0, \frac{2-3\alpha}{2(1-\alpha)}\right\}$. \square

In summary we now obtain the claim for ν U α H: Let $\delta = \alpha(1 - \epsilon)$, $0 < \epsilon < 1$. It suffices to prove the statement for ϵ sufficiently small. Let β such that $\gamma(\alpha, \beta) = \delta$, i.e. $\beta = 1 - \frac{\alpha}{2(1-\alpha)}\epsilon$. Choosing ϵ sufficiently small we can ensure that $\beta > \frac{2-3\alpha}{2(1-\alpha)}$ which is required to apply Lemma 5.2.

For such choice of ϵ and β , Lemma 5.2 implies that for any Borel set B with $h^\delta(B) = 0$,

$$(5.5) \quad \kappa(B) = \int \mu_{\lambda, \text{sing}}(B \cap T_{\mu; \infty}^\beta) d\nu(\lambda) \leq \int \mu_{\lambda, \text{sing}}(T_{\mu; \infty}^1) d\nu(\lambda) = 0 .$$

Applying Proposition 2.2 and 2.1, $\mu_{\lambda, \text{sing}}(T_{\mu; \infty}^1) = 0$ for $\lambda \neq 0$, which by continuity of ν implies the last equality in (5.5).

Step 2: Let $0 < \alpha < 1$ and $\delta < \alpha$. If ν is α c, then by Theorem 4.3 given $\epsilon > 0$ there are measures $\nu_1 \perp \nu_2$, $\nu = \nu_1 + \nu_2$, such that ν_1 is U α H and $\nu_2(\mathbb{R}) < \epsilon$. Let $B \subseteq \mathbb{R}$ be a Borel set with $h^\delta(B) = 0$. Then, $\int \mu_\lambda(B) d\nu_1(\lambda) = 0$ by step 1, whence

$$\kappa(B) = \int \mu_\lambda(B) d\nu_2(\lambda) < \epsilon .$$

An analogous argument shows that κ is absolutely continuous if $\alpha = 1$, which concludes the proof of Theorem 1.2.

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